Shumakovitch

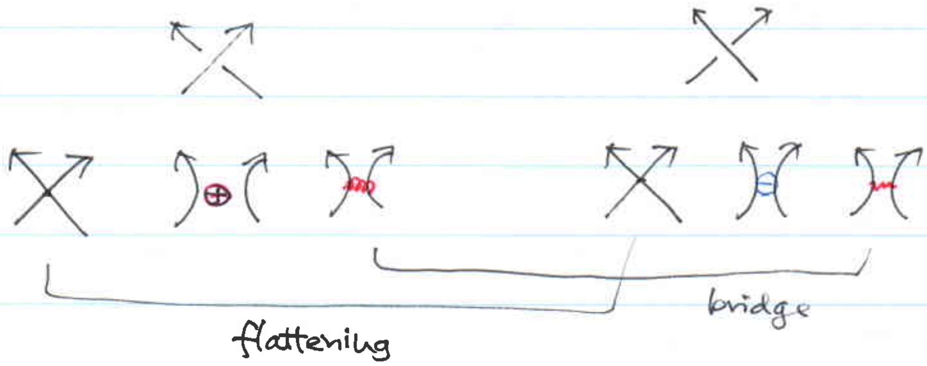
Fix $N>0$

$$-q^{-N} P_N(\times) + q^N P_N(\times) = (q-q^{-1}) P_N(\asymp)$$

$$P_N(\bigcirc) = [N]_q = \frac{q^N - q^{-N}}{q-q^{-1}}$$

State sum formula for P_N following [MOY]

each crossing $\rightsquigarrow$ 3 possible resolutions



Γ is a choice of a resolution at every crossing of a diagram D of a link L, so $|\Gamma| = 3^{\underbrace{\#\text{crossings}}_{\geq k}}$

Let D_Γ be the result of applying Γ to D

$\uparrow$ collection of oriented planar curves $\qquad |D_\Gamma| = \#\{\text{comp. of } D_\Gamma\}$

Def. Γ is said to be **admissible** if all comp. of D_Γ are simple

$\forall \alpha_i$: comp. of D_Γ let $\mathrm{rot}(\alpha_i)$ be its rotation #

Exercise $\quad \Gamma$ is admissible $\Rightarrow \mathrm{rot}(\alpha_i) = \pm 1$

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Def. $g(\Gamma) = \# \oplus - \# \ominus$

Palette $\mathcal{N} = \{-N+1, -N+3, \cdots, N-1\}$ $\quad |\mathcal{N}| = N$

Def. coloring C of D_Γ is a map $\{\text{comp. of } D_\Gamma\} \longrightarrow \mathcal{N}$

Def. C is admissible if at & the colors of two branches are different.

Def. $$rot(C) = \sum_{i=1}^{|D_\Gamma|} rot(\alpha_i)\, C(\alpha_i)$$

$$\tau(C) = \#\{ \ \mid a<b\} - \#\{ \ \mid a>b\}$$

Exercise

$$P_N(L) = q^{Nw(D)} \sum_{\substack{\Gamma, C \\ \text{admissible}}} (-1)^{k+g(\Gamma)} q^{-g(\Gamma)+rot(C)+\tau(C)}$$

$w(D) = \text{writhe} = k_+ - k_-$

$g(\Gamma) = \# \oplus - \# \ominus$

$i(\Gamma, C) = -k + g(\Gamma)$

$j(\Gamma, C) = -g(\Gamma) + rot(C) + \tau(C)$

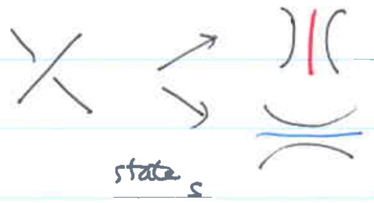
$$P_N(L) = q^{Nw(D)} \sum_{\Gamma, C} (-1)^{i(\Gamma,C)} q^{j(\Gamma,C)}$$

Def. Let $\bigoplus C^N_{ij}(D)$ bigraded $\mathbb{Z}$-module with C^N_{ij} freely generated by all admissible Γ, C

s.t. $i(\Gamma, C) = i$, $j(\Gamma, C) = j$

complex??

Remember Khovanov



state s

$g(s) = \# | - \# —$

enhanced state

choice of $\pm$ on each resolved circle

$\Rightarrow$ Khovanov complex $\bigoplus C^{ij}(D)$

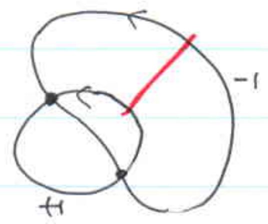
freely generated by enh. states s.t. $i(s) = i$, $j(s) = j$

Lemma $C^{N=2}(D)[2w(D)] \cong C(D)$ as bigraded modules

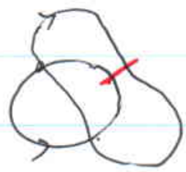
(proof) there is a bijection between generators.

$= 2 \quad = \mathcal{N} = \{\pm 1\}$

$sl_2 \Rightarrow$ Jones

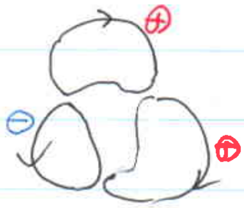


$\rightsquigarrow$ reverse ori. of all circles of the -1



$\rightsquigarrow$ cut all

according to the original picture



Jones $\Rightarrow sl_2$

inverse the construction

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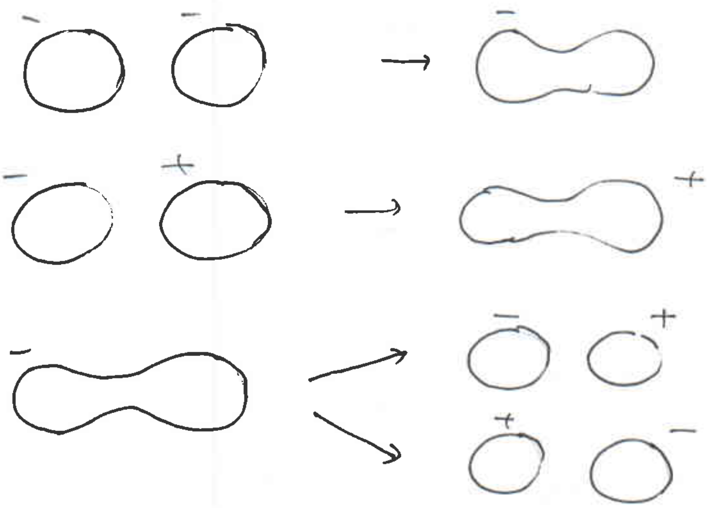
Khovanov differential $C^{i,j}(D)$ induces a differential on $C^{N=2}_{i,j}(D)$

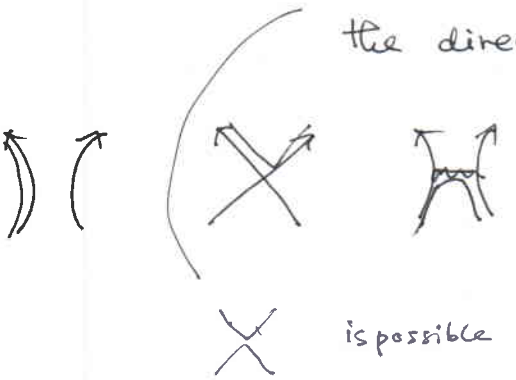
Th This differential can be generalised to $C^N(D)$ for all $N \geq 2$.

Idea: look at differential for $N=2$



We do
1) fusion
2) splitting
3) change of sign (switching)

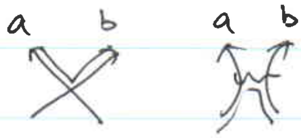
Def. A Kauffman cycle γ in D_Γ is a collection of edges of D_Γ that form an almost closed curve that changes the direction at each 



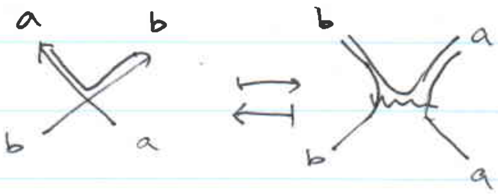
is possible

Def. C : admissible

A K cycle γ is said to be switchable with colors from C if it is colored with at most 2 colors a and b s.t. $|a-b|=2$



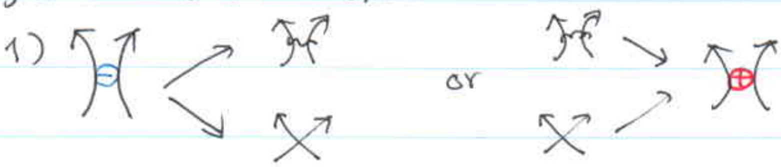
Let γ be switchable. Denote by $(\Gamma_\gamma, C_\gamma)$ the result of exchanging of two colors on γ.



Rem If we only have one color a change it into $a \pm 2$

Lemma If γ is switchable, then $j(\Gamma, C) = j(\Gamma_\gamma, C_\gamma) \pm 2$

Ingredient of the differential



Also

2) cycle fusion / splitting

3) switching of K cycles (before or after)

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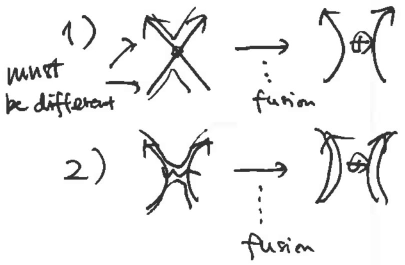
Assume we are in the case



3) the same but splitting

switchable K cycles have signs.

1) each edge of D has an index winding number of D w.r.t. the region to the right of this edge

2) edges of γ with even/odd index have the same colors

3) $s(\gamma) = \text{sign}\left(\text{color with odd ind.} - \text{color with even ind.} \right)$

4) change ori. at all edges with say, odd, ind

all K cycles are oriented now

Let $w(\gamma) = \text{rot}(\gamma) = I$,

5) $\text{sign}(\gamma) = s(\gamma) w(\gamma)$

If a cycle with $s<0$ has $w =$ parity

switch it and face the result

R1 $C(\ ) \underset{\text{homotopic}}{\cong} C(\ )\{N\}$

R2 66% OK

R3 complicated.